Exercise Sheet #1

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P1. Show that $J^*(\mathbb{Q} \cap [0,1]) = J^*([0,1] \backslash \mathbb{Q}) = 1$, and $J_*(\mathbb{Q} \cap [0,1]) = J_*([0,1] \backslash \mathbb{Q}) = 0$.

P2. Let $U \subseteq \mathbb{R}$ be an open set. Show that U can be written as a disjoint union of countably many open intervals.

P3. Let $U = \left\{(x,y) : x^2 + y^2 < 1\right\} \subseteq \mathbb{R}^2$ be the open unit disk. Show that U cannot be expressed as a disjoint union of countably many open boxes.

P4. Give an example to show that the statement

$$\lambda^*(E) = \sup_{U \subset E, U \text{ open}} \lambda^*(U)$$

is false.

P5. (Area interpretation of the Riemann integral). Let $[a,b]$ be an interval, and let $f : [a,b] \to \mathbf{R}_+ := [0,\infty)$ be a bounded function. Show that f is Riemann integrable if and only if the set $E_+ := \{(x,t) : x \in [a,b]; 0 \le t \le f(x)\}$ is Jordan measurable in $\mathbf{R}^2$, in which case one has

$$\int_a^b f(x)dx = m^2\left(E_+\right).$$

where m^2 denotes two-dimensional Jordan measure.
Remark: For $f : [a,b] \to \mathbb{R}$ not necessarily positive, one can prove that f is Riemann integrable if and only if the sets E_+ and $E_- := \{(x,t) : x \in [a,b]; f(x) \le t \le 0\}$ are both Jordan measurable in $\mathbf{R}^2$, in which case one has $\int_a^b f(x)dx = m^2\left(E_+\right) - m^2\left(E_-\right).$

P6. (**Homework**) Let $U \subseteq \mathbb{R}^d$ be an open set. Show that U can be written as a disjoint union of countably many half-open boxes (i.e., sets of the form $B = \prod_{i=1}^d [a_i, b_i)\Big)$.